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Inert Sets and the Lie Algebra Associated to a Group

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Independently, J. Labute (*Trans. Amer. Math. Soc.* **288** (1985), 51–57) has given a sufficient condition for the Lie algebra associated with a discrete group's lower central series to have a certain presentation, and S. Halperin and J.-M. Lemaire (*Math. Scand.*, in press) have discussed the conditions under which some homogeneous elements in a graded algebra constitute an "inert set." As it turns out, these conditions essentially coincide. As a consequence, various theorems about graded algebras may now be translated into results into group theory. A nontrivial application to link groups is offered.

1. THE LABUTE–HALPERIN–LEMAIRE CONDITION

In [11] Labute considered a finitely presented group

$$G = \langle x_1, ..., x_g \mid r_1, ..., r_t \rangle \tag{1}$$

and asked the following question. Let $\mathscr{L}G$ denote the graded Lie algebra (over $\mathbb{Z}$) obtained from the lower central series of G and let $F = \langle x_1, ..., x_g \rangle$ be the free group which surjects to G. Let

$$F = F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$$

be the lower central series for F and let $m_i = \sup\{j \mid r_i \in F_j\}$ be the *weight* of the ith relation r_i. If ρ_i denotes the image of r_i in $(\mathscr{L}F)_{m_i} = F_{m_i}/F_{m_i+1}$, let $\mathscr{M}G$ denote the quotient of $\mathscr{L}F$ by the Lie ideal J which $\rho = \{\rho_1, ..., \rho_t\}$ generates. (Note: $\mathscr{M}G$ depends upon the specific presentation adopted for G.) The question is, when is there an isomorphism $\mathscr{L}G \approx \mathscr{M}G$?

As a partial answer to this question, Labute proved in [11, Theorems 1 and 2] the following result.

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THEOREM 1.1. *Suppose that*

(2a) *the quotient Lie algebra $\mathcal{M}G$ is a free $\mathbb{Z}$-module; and*

(2b) *$J/[J, J]$ is a free $\mathcal{U}(\mathcal{M}G)$-module via the adjoint action, with the images $\{\bar{\rho}_1, ..., \bar{\rho}_t\}$ of the $\{\rho_i\}$ in $J/[J, J]$ comprising a basis.*

Then $\mathcal{L}G \approx \mathcal{M}G$ as graded Lie algebras, and the universal enveloping algebra $\mathcal{U}(\mathcal{L}G)$ coincides with the associated graded algebra $\mathrm{gr}(\mathbb{Z}G)$ of the integral group ring of G filtered by powers of its augmentation ideal.

DEFINITION 1.2. A set $r = \{r_1, ..., r_t\}$ of elements in the free group $F = \langle x_1, ..., x_g \rangle$ will be called a *mild* set if and only if both (2a) and (2b) are satisfied. The quotient group $G = F/\langle r_1, ..., r_t \rangle$, the angle brackets here denoting normal closure, will be called a *mild* group.

The images $\{\rho_1, ..., \rho_t\}$ of $\{r_1, ..., r_t\}$ are homogeneous in the free graded Lie algebra $\mathcal{L}F$. Because Definition 1.2 depends only upon $\{\rho_1, ..., \rho_t\}$ we might expect that the condition also plays a role in the theory of graded Lie algebras. We digress next into a brief discussion of graded algebras. Without saying so explicitly we will sometimes identify a graded Lie algebra L with the Lie algebra of primitives in its universal enveloping algebra $\mathcal{U}(L)$ [12]. With this convention, we may think of $\rho \subseteq \mathcal{L}F$ as a subset of $\mathcal{U}(\mathcal{L}F)$.

Let p denote any prime or zero, and let $\mathbb{Z}_p$ denote the prime field of characteristic p. If $A = \bigoplus_{n=0}^{\infty} A_n$ is a connected graded $\mathbb{Z}$-algebra, then there is an obvious homomorphism $A \to A \otimes \mathbb{Z}_p$, where $A \otimes \mathbb{Z}_p$ is a connected graded $\mathbb{Z}_p$-algebra. If A is locally finite (i.e., each A_n is a finitely generated $\mathbb{Z}$-module), then A is a free $\mathbb{Z}$-module if and only if all of the $A \otimes \mathbb{Z}_p$'s have the same Hilbert series (see [3] for definitions).

When $\{r_1, ..., r_t\}$ is a mild set, we shall see next that the corresponding homogeneous $\{\rho_1, ..., \rho_t\}$ form what is called an "inert" or "strongly free" set (these two terms are interchangeable) in the free associative graded algebra on g generators. Up to now, the concept of an inert set has been defined only for graded algebras over a field. We recall that definition and extend it to graded $\mathbb{Z}$-algebras now.

DEFINITION 1.3. Let k be a field. Given a set of homogeneous elements $\{\rho_1, ..., \rho_t\}$ in an associative connected graded k-algebra A, let $q: A \to B = A/\langle \rho_1, ..., \rho_t \rangle$ denote the quotient map. The set $\{\rho_1, ..., \rho_t\}$ is *inert* if and only if the induced map $q_*: \mathrm{Tor}_m^A(k, k) \to \mathrm{Tor}_m^B(k, k)$ is injective for $m = 2$ but isomorphic for $m > 2$. A set $\{\rho_1, ..., \rho_t\}$ of homogeneous elements in a connected graded $\mathbb{Z}$-algebra A is *inert* if and only if, for every prime p, its image in $A \otimes \mathbb{Z}_p$ forms an inert set in $A \otimes \mathbb{Z}_p$.

The connection with mild sets may be made at once via the following theorem of Halperin and Lemaire.

THEOREM 1.4. *Let $\rho = \{\rho_1, ..., \rho_t\}$ be a set of homogeneous elements in a connected graded Lie k-algebra L, and let J be the Lie ideal ρ generates. Then ρ is an inert set in $\mathscr{U}(L)$ if and only if*

(3a) *ρ is a minimal set of generators for the Lie ideal J;*

(3b) *J is a free Lie subalgebra of L; and*

(3c) *$J/[J, J]$ is a free $\mathscr{U}(L/J)$-module via the adjoint action.*

Proof. See [8, Theorem 3.3]. Halperin and Lemaire actually prove this under the restriction that $\mathrm{char}(k) = 0$, but the proof nowhere uses that restriction.

THEOREM 1.5. *Let $r = \{r_1, ..., r_t\}$ be any subset of the free group $F = \langle x_1, ..., x_g\rangle$. Then the corresponding set $\rho = \{\rho_1, ..., \rho_t\}$ of homogeneous elements in $\mathscr{U}(\mathscr{L}F) \approx \mathbb{Z}\langle \xi_1, ..., \xi_g\rangle$ is inert if and only if r is a mild set.*

Proof. If r is mild, let $\rho(p)$ denote the image of ρ under the homomorphism $\mathscr{U}(\mathscr{L}F) \to \mathscr{U}(\mathscr{L}F) \otimes \mathbb{Z}_p$, and let $J(p)$ be the Lie ideal generated by $\rho(p)$. Condition (2) immediately yields both (3a) and (3c). Because $(\mathscr{L}F) \otimes \mathbb{Z}_p$ is a free Lie algebra, condition (3b) is automatically satisfied [12].

Conversely, if ρ is inert, then each $\rho(p)$ is inert in $\mathscr{U}(\mathscr{L}F) \otimes \mathbb{Z}_p$. For each p, $J(p)/[J(p), J(p)]$ is a free $\mathscr{U}(\mathscr{L}F/J) \otimes \mathbb{Z}_p$-module, with $\rho(p)$ as a basis. A simple Hilbert series argument based on the Poincaré–Birkhoff–Witt theorem shows that $\mathscr{L}F/J$ and $J/[J, J]$ must be free $\mathbb{Z}$-modules. In particular, (2a) holds, and if

$$\lambda\colon \mathscr{U}(\mathscr{L}F/J) \otimes \left(\bigoplus_{\rho_i \in \rho} \mathbb{Z}\right) \to J/[J, J]$$

is the surjection defined by the adjoint action on ρ, then for every p we have $\ker(\lambda) \otimes \mathbb{Z}_p = 0$. Thus $\ker(\lambda) = 0$, i.e., (2b) holds too.

The usefulness of Theorem 1.5 lies in the fact that inert sets have been rather carefully studied, with [3, 8] being the major references. In particular, from Section 2 of [3] we may extract

THEOREM 1.6. *Let k be a field and let $\rho = \{\rho_1, ..., \rho_t\}$ be a set of homogeneous elements in the free associative graded algebra $C = k\langle \xi_1, ..., \xi_g\rangle$. Let $I = \langle \rho_1, ..., \rho_t\rangle$ denote the two-sided ideal of C generated by ρ and put $A = C/I$.*

The following are equivalent:

(4a) *ρ is inert in C;*

(4b) $\mathrm{gl.\,dim}(A) \leqslant 2$;

(4c) *the Hilbert series of A is*

$$A(z)=\left[1-\sum_{i=1}^{g} z^{|\xi_i|}+\sum_{j=1}^{t} z^{|\rho_j|}\right]^{-1}.$$

From this there follows a characterization of inert sets in a free $\mathbb{Z}$-algebra.

COROLLARY 1.7. *Let $\rho=\{\rho_1, ..., \rho_t\}$ be a set of homogeneous positive-dimensional elements in the free connected graded $\mathbb{Z}$-algebra $C=\mathbb{Z}\langle\xi_1, ..., \xi_g\rangle$. Let $I=\langle\rho_1, ..., \rho_t\rangle$ denote the two-sided ideal generated by ρ, and assume that ρ is a minimal set of generators for I. With $A=C/I$, ρ is inert if and only if*

(5a) *A is a free $\mathbb{Z}$-module; and*

(5b) $\operatorname{Ext}_A^3(\mathbb{Z}, M)=0$ *for all graded right A-modules M.*

Proof. Each $\rho_j \in C$ may be written uniquely as

$$\rho_j=\sum_{i=1}^{g} \xi_i\sigma_{ij}, \qquad \sigma_{ij}\in C.$$

Consider the sequence of right A-module homomorphisms

$$0 \leftarrow \mathbb{Z} \leftarrow A \xleftarrow{\delta_1} \bigoplus_{i=1}^{g} A \xleftarrow{\delta_2} \bigoplus_{j=1}^{t} A \leftarrow 0, \tag{6}$$

in which $\delta_1(a_1, ..., a_g)=\xi_1 a_1+\cdots+\xi_g a_g$ and $\delta_2(b_1, ..., b_t)=(\sum_{j=1}^{t}\bar\sigma_{1j}b_j, ..., \sum_{j=1}^{t}\bar\sigma_{gj}b_j)$, the overbars signifying the image under the projection $C\to A$. The sequence (6) is exact, except possibly at the term $\bigoplus_{j=1}^{t} A$.

If ρ is inert, then by (4c) the Hilbert series of $A\otimes\mathbb{Z}_p$ does not depend upon p, hence A must be a free $\mathbb{Z}$-module. Supposing A to be a free $\mathbb{Z}$-module, we now demonstrate the equivalence between condition (5b) and the inertness of ρ. Because A is connected and free over $\mathbb{Z}$, $\operatorname{Ext}_A^3(\mathbb{Z}, M)$ will vanish for all graded A-modules M if and only if (6) is exact. By the universal coefficient theorem, (6) is exact if and only if its homology with coefficients in each $\mathbb{Z}_p$ is zero. By (4b), the latter happens if and only if ρ is inert.

COROLLARY 1.8. *Let G be a mild group and let $A=\operatorname{gr}(\mathbb{Z}G)$ be the associated graded algebra to the group ring $\mathbb{Z}G$. Then A is free over $\mathbb{Z}$ and* $\operatorname{Ext}_A^3(\mathbb{Z}, M)=0$ *for all graded right A-modules M.*

Proof. By Theorem 1.1, A coincides with $\mathscr{U}(\mathscr{M}G)$ and, by Theorem 1.5,

A equals the quotient of a free associative $\mathbb{Z}$-algebra by an inert set. The result follows from Corollary 1.7.

In view of Corollary 1.8 and [5, pp. 184–185], we might say that a mild group G has "graded cohomological dimension two." The author originally hoped that a spectral sequence similar to that of [1, Theorem 1.2] could be used to deduce that G has actual cohomological dimension two, but this approach does not work. The next section includes some examples of mild groups whose cohomological dimension exceeds two.

2. Examples

Various sufficient conditions are known for a set of homogeneous elements in a free algebra to be inert. By applying these conditions we can demonstrate the existence of a wide range of mild groups.

The single most successful criterion for inertness so far discovered is developed in Section 3 of [3]. The criterion involves the "high terms" of the elements making up the set under consideration. If α is any nonzero homogeneous element of degree d in the free associative k-algebra $C = k\langle \xi_1, ..., \xi_g \rangle$, let $\{w_1, ..., w_m\}$ be a complete list of the monomials in $\{\xi_1, ..., \xi_g\}$ which have degree d. Write α (uniquely) as a linear combination

$$\alpha = \sum_{j=1}^{m} c_j w_j, \qquad c_j \in k.$$

Choose an arbitrary total ordering on $\{\xi_1, ..., \xi_g\}$ and note that $\{w_1, ..., w_m\}$ inherits the lexicographic order. The *high term* of α is the highest w_j (with respect to the lexicographic order) for which $c_j \neq 0$.

THEOREM 2.1. *Let $\{\alpha_1, ..., \alpha_t\}$ be any set of nonzero homogeneous elements in the free assciative k-algebra $C = k\langle \xi_1, ..., \xi_g \rangle$. Fix an ordering of $\{\xi_1, ..., \xi_g\}$ and let w_i be the high term of α_i. Suppose that*

(7a) *no w_i is a submonomial of any w_j for $i \neq j$, i.e., $w_j = uw_iv$ cannot occur; and*

(7b) *no w_i "overlaps with" any w_j, i.e., $w_i = uv$ and $w_j = vw$ cannot occur unless $v = 1$ or $u = w = 1$.*

Then $\{\alpha_1, ..., \alpha_t\}$ is inert in C.

Proof. See [3, Theorem 3.2].

When $\rho=\{\rho_1, ..., \rho_t\} \subseteq \mathscr{U}(\mathscr{L}F)$ is the set arising from a set of relations $r=\{r_1, ..., r_t\}$ in a free group F, the ρ_i's will be *Lie elements*, i.e., they will equal linear combinations of repeated commutators of the generators. For example,

$$r_1=[x_1, x_3][x_2, x_3]^2[x_1, [x_2, x_3]]$$

yields

$$\rho_1=[\xi_1, \xi_3]+2[\xi_2, \xi_3]=\xi_1\xi_3-\xi_3\xi_1+2\xi_2\xi_3-2\xi_3\xi_2,$$

whose high term (taking $\xi_1>\xi_2>\xi_3$) is $\xi_1\xi_3$. Likewise

$$r_2=[x_1, [x_3, x_2]]$$

leads to

$$\begin{aligned}\rho_2=[\xi_1, [\xi_3, \xi_2]]&=\xi_1(\xi_3\xi_2-\xi_2\xi_3)-(\xi_3\xi_2-\xi_2\xi_3)\xi_1\\ &=-\xi_1\xi_2\xi_3+\xi_1\xi_3\xi_2+\xi_2\xi_3\xi_1-\xi_3\xi_2\xi_1,\end{aligned}$$

with high term $\xi_1\xi_2\xi_3$. Note that the doubleton $\{\xi_1\xi_3, \xi_1\xi_2\xi_3\}$ does satisfy the condition (7), so $\{\rho_1, \rho_2\}$ is inert, for any k.

Using criterion (7) it becomes virtual to check that each of the four examples on p. 52 of [11] is an inert set. For examples 2, 3, and 4 of [11], take $\xi_1>\cdots>\xi_N$, but for example 1 order $\{\xi_i\}$ so that $\xi_i>\xi_j$ whenever i is even and j is odd.

Suppose a list of weights $\{m_1, ..., m_t\}$ is specified in advance. Does there exist a mild group whose presentation is given by (1), where r_i has weight m_i? A partial answer to this question is given by

THEOREM 2.2. *The answer to the above question is "yes" if there exist polynomials $u(z)$ and $v(z)$, having nonnegative integer coefficients and zero constant terms, such that the inequality*

$$(1-u(z))(1-v(z))\geqslant 1-gz+\sum_{j=1}^{t} z^{m_j}$$

holds coefficientwise.

Proof. See [2, Proposition 5.6] and apply Theorem 1.5.

For example, if $g=2$ and $t=3$ with $m_1=m_2=5$ and $m_3=6$, the fact that

$$(1-z-z^3-z^4)(1-z-z^2)=1-2z+2z^5+z^6$$

guarantees that a mild set $r=\{r_1, r_2, r_3\}$ exists in $F=\langle x, y\rangle$ with r_i having weight m_i. In this instance the set

$$r_1=[[x,[x,y]],[x,y]]$$
$$r_2=[[[x,[x,y]],y],y]$$
$$r_3=[[[x,[x,y]],y],[x,y]]$$

works. The set of high terms (taking $\xi_x>\xi_y$) of the corresponding $\{\rho_1, \rho_2, \rho_3\}$ is $\{\xi_x^2\xi_y\xi_x\xi_y, \xi_x^2\xi_y^3, \xi_x^2\xi_y^2\xi_x\xi_y\}$, which may easily be seen to satisfy (7).

Following [2, Proposition 5.8], one can find arbitrarily large inert sets of Lie elements in $\mathbb{Z}\langle x, y\rangle$ in which all elements share the same degree m. Consequently there exist two-generator mild groups having arbitrarily large negative deficiencies.

As a final type of example of mild groups, consider

LEMMA 2.3. *Suppose* (1) *is a presentation for G, and that $H_1(G)$ is a free abelian group of rank $g-t$. Then G is a mild group, and $\mathscr{L}G$ is a free Lie algebra on $g-t$ generators.*

Proof. Let $\bar{r}_j$ denote the image of r_j in $F/[F,F]\approx\bigoplus_{i=1}^{g}\mathbb{Z}$. We have an exact sequence of abelian groups

$$\bigoplus_{j=1}^{t}\mathbb{Z}\xrightarrow{\ v\ }\bigoplus_{i=1}^{g}\mathbb{Z}\longrightarrow H_1(G)\longrightarrow 0,$$

where $v(a_1, ..., a_t)=a_1\bar{r}_1+\cdots+a_t\bar{r}_t$. Because $H_1(G)$ is free abelian of rank $g-t$, $\ker(v)=0$ and the set $\{v(\bar{r}_1), ..., v(\bar{r}_t)\}$ may be extended to a $\mathbb{Z}$-basis $\{\xi'_1, ..., \xi'_g\}$ for $F/[F,F]$.

When $\mathscr{L}F$ is described in terms of this basis, we see that $\mathscr{L}F$ is the free Lie algebra on $\{\xi'_1, ..., \xi'_g\}$ and that $\rho_i=\xi'_i$ for $1\leqslant i\leqslant t$. The set ρ is obviously inert, the quotient $\mathscr{M}G=\mathscr{L}F/J$ being isomorphic to the free Lie algebra on $\{\xi'_{t+1}, ..., \xi'_g\}$.

COROLLARY 2.4. *Any tame knot group G is a mild group, and* $[[G,G],G]=[G,G]$.

Proof. If a knot's projection has n strands, then the Wirtinger presentation for its group has n generators and $n-1$ relations. Furthermore, its first homology group is $\mathbb{Z}$. By the previous lemma, if $\{G_i\}$ is the lower central series for G, then $G_2/G_3=(\mathscr{L}G)_2\approx(\mathbb{L}\langle\xi_1\rangle)_2=0$.

COROLLARY 2.5. *The binary icosahedral group I^* and the group $E=\langle b, c\mid b^{-2}cbc, c^{-6}bcb\rangle$ are mild groups having cohomological dimension ∞ and* 3, *respectively.*

Proof. A presentation for I^* is

$$I^* = \langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle,$$

which is mild by Lemma 2.3 with $g = t = 3$ because $H_1(I^*; \mathbb{Z}) = 0$. Being a finite group, I^* has infinite cohomological dimension (see Corollary 2.5, p. 187 of [5]). A second presentation for E is

$$E = \langle a, b, c \mid a^2 = b^3 = c^7 = abc \rangle,$$

which shows it to be the fundamental group of a closed Seifert fibered 3-manifold S. There are three singular fibers for S, with indices 2, 3, and 7. The universal cover for S is $\widetilde{SL(2, \mathbb{R})} \approx \mathbb{R}^3$, hence S is a $K(E, 1)$. Since $H^3(E; \mathbb{Z}) = H^3(S; \mathbb{Z}) = \mathbb{Z}$, $\operatorname{c.dim}(E) \geqslant 3$. By [6], E has cohomological dimension precisely three.

3. Application to Links

In this section we will prove, for a certain class of links in S^3, that the fundamental groups of their complements are mild groups.

Link groups, that is, $\pi_1(S^3 - L)$ for tame links $L \subseteq S^3$, are an important source of finitely presented groups. Sometimes these groups are mild groups. For instance, examples 1, 2, and 3 of [11] correspond respectively to a chain, to a "key ring," and to the Borromean rings. These links are in turn examples of what we will call "pure braid links."

Definition 3.1. A *pure braid link* (on n strands) is the link of n unknots in S^3 which is obtained from a pure braid by identifying the top and bottom of each strand.

Note. A pure braid is an element of the braid group B_n which lies in the kernel of the canonical homomorphism from B_n to the symmetric group S_n.

The advantage of restricting ourselves to pure braid links is that their link groups have fairly understandable presentations with n generators and $n - 1$ relations, where n equals the number of components in the link. This restriction is not without cost. It excludes, among others, the Whitehead link.

To any n-component link we may associate its *linking matrix* $\gamma = (\gamma_{ij})$, a symmetric integer matrix with zeroes down the diagonal in which γ_{ij} equals the linking number of the ith component with the jth component.

If L is a pure braid link, an easy inductive argument based upon Theorem N8 of [9, pp. 173–174] shows

LEMMA 3.2. *Let L be an n-strand pure braid link. A presentation for $G=\pi_1(S^3-L)$ is*

$$G \approx \langle x_1, ..., x_n \mid r_1, ..., r_{n-1} \rangle,$$

where

$$r_i = [x_i, \tau_i] \tag{8}$$

for some τ_i. Furthermore,

$$\tau_i \equiv x_i^{\delta_i} \prod_{j=1}^{n} x_j^{\gamma_{ij}} \qquad (\text{mod } [F, F])$$

for some δ_i, where $F = \langle x_1, ..., x_n \rangle$ is the free group.

It is worth noting that any zero-diagonal symmetric integer matrix γ can be realized as the linking matrix of a pure braid link. To see this, note that for $i<j$ the pure braid $\alpha_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2}(\sigma_{j-1})^{-2}(\sigma_{j-2})^{-1} \cdots (\sigma_i)^{-1}$ has the effect of increasing γ_{ij} and γ_{ji} by one but of leaving the other linking numbers unchanged. Any γ can therefore be attained through a suitable product of α_{ij}'s.

An immediate consequence of Lemma 3.2 is

LEMMA 3.3. *Let L be an n-component pure braid link and let $\gamma = (\gamma_{ij})$ be its linking matrix. Suppose that, for each i, some γ_{ij} is nonzero (equivalently, suppose that every row of γ is nonzero). Then each relation r_i in* (8) *has weight two, and the corresponding relations in $\mathscr{U}(\mathscr{L}F)$ are described by*

$$\rho_i = \sum_{j=1}^{n} \gamma_{ij} [\xi_i, \xi_j]. \tag{9}$$

By Theorem 1.5, if $\rho = \{\rho_1, ..., \rho_{n-1}\}$ is an inert set, then $G = \pi_1(S^3 - L)$ will be a mild group. While the inertness of ρ does depend only upon the matrix γ_{ij}, the sufficient condition we will obtain can more easily be described in terms of a certain diagram than in terms of a matrix.

DEFINITION 3.4. Let $\beta = (\beta_{ij})$ be any integer $n \times n$ matrix having $\beta_{ii} = 0$ and $\beta_{ij} = \beta_{ji}$. The *diagram associated to* β will be a complete edge-weighted graph on n vertices labeled v_1 through v_n. For each pair of distinct integers (i, j), the edge joining v_i and v_j is assigned a weight of β_{ij}. Such a diagram is *connected* mod p if and only if there exists a spanning subtree consisting of edges whose weights are incongruent to zero, modulo p. If γ is the linking matrix of an n-component link L, the diagram associated to γ is called the *linking diagram* for L.

Note. The linking diagram is closely related to the perhaps more familiar Dehn plumbing diagram used in the theory of surgery on three-manifolds. The differences are that the linking diagram's edges are weighted by linking numbers, which are often assumed to be 0 (omit the edge) or ± 1 (include the edge) in the plumbing diagram, and that the plumbing diagram's vertices are labeled with rational numbers to show how the removed solid tori should be reattached (cf. [7, p. 34]).

PROPOSITION 3.5. *Fix a prime* (or *zero*) p. *Let* D *be the diagram associated to an integer matrix* $\beta = (\beta_{ij})$ *having* $\beta_{ii} = 0$ *and* $\beta_{ij} = \beta_{ji}$. *For* $1 \leqslant i \leqslant n$ *let*

$$\rho_i = \sum_{j=1}^{n} \beta_{ij} [\xi_i, \xi_j] \tag{10}$$

in $\mathbb{Z}\langle \xi_1, ..., \xi_n \rangle$. *If* D *is connected* mod p, *then the image of* $\rho = \{\rho_1, ..., \rho_{n-1}\}$ *in* $\mathbb{Z}_p\langle \xi_1, ..., \xi_n \rangle$ *is inert.*

Proof. We will show that there exists an ordering of $\{\xi_1, ..., \xi_n\}$ with respect to which the set of high terms of $\{\rho_1, ..., \rho_{n-1}\}$ satisfies condition (7).

To define this ordering, view the edge weights on D as elements of $\mathbb{Z}_p$ and pick a spanning subtree D' in which each edge has nonzero weight. We will partition the vertex set $\{v_1, ..., v_n\}$ into two subsets called the "high" and the "low" vertices.

To accomplish this, first note that there is a permutation $\mu \in S_n$ such that $\mu(n) = n$ and for each $j \geqslant 1$ there exists a subtree of D' whose vertex set is $\{v_{\mu(j)}, v_{\mu(j+1)}, ..., v_{\mu(n)}\}$. To see this, just take $v_{\mu(n)} = v_n$; for $v_{\mu(n-1)}$ pick any vertex which is joined by an edge of D' to $v_{\mu(n)}$; for $v_{\mu(n-2)}$ take any vertex which has an edge of D' joining it to $v_{\mu(n)}$ or to $v_{\mu(n-1)}$, etc.

Having found μ, we determine the high–low partition inductively, as follows. Assign $v_n = v_{\mu(n)}$ to be "high." Having assigned $v_{\mu(j)}$ for $n \geqslant j > m$, define $v_{\mu(m)}$ to be "high" if $\beta_{\mu(j),\mu(m)} = 0$ in $\mathbb{Z}_p$ for all $j > m$ such that $v_{\mu(j)}$ is high, and define it to be "low" otherwise. Note that $\beta_{ij} = 0$ in $\mathbb{Z}_p$ whenever v_i and v_j are both high.

Once all vertices have been assigned, order $\{v_1, ..., v_n\}$ such that $v_{\mu(i)} > v_{\mu(j)}$ if $v_{\mu(i)}$ is high and $v_{\mu(j)}$ is low, but when both are high or both are low put $v_{\mu(i)} > v_{\mu(j)}$ precisely when $i > j$. Let the generators $\{\xi_1, ..., \xi_n\}$ of $\mathbb{Z}_p\langle \xi_1, ..., \xi_n \rangle$ be ordered so that $\xi_i > \xi_j$ whenever $v_i > v_j$.

We claim that the set of high terms of $\{\rho_{\mu(1)}, ..., \rho_{\mu(n-1)}\}$, with respect to the ordering induced by $>$, satisfies (9). To see this, consider two cases. If $v_{\mu(m)}$ is high, then over $\mathbb{Z}_p$ the expression $\rho_{\mu(m)}$ equals a linear combination of commutators of $\xi_{\mu(m)}$ with low ξ_i's. Furthermore, by our choice of μ, $\beta_{\mu(m),\mu(j)} \neq 0$ in $\mathbb{Z}_p$ for some $j > m$. The high term of $\rho_{\mu(m)}$ therefore has the form $\xi_{\mu(m)}\xi_{\mu(j)}$ for some j such that $v_{\mu(j)}$ is low but $j > m$.

If instead $v_{\mu(m)}$ is low, this means that $\beta_{\mu(j),\mu(m)} \neq 0$ in $\mathbb{Z}_p$ for some high $v_{\mu(j)}$, where $j > m$. Deduce that the high term of $\rho_{\mu(m)}$ has the form $\xi_{\mu(j)}\xi_{\mu(m)}$, where $v_{\mu(j)}$ is high and $j > m$.

Summarizing, the high term of each ρ_i, $1 \leqslant i \leqslant n-1$, has the form

$$w_i = \xi_{s(i)}\xi_{t(i)},$$

where one of $\{s(i), t(i)\}$ equals i and the other, call it i', satisfies $\mu^{-1}(i') > \mu^{-1}(i)$. Consequently $\{w_1, ..., w_{n-1}\}$ are all distinct. Furthermore, $v_{s(i)}$ is always high and $v_{t(i)}$ is always low, so that two w_i's cannot overlap as in (7b). Condition (7) is met, and consequently ρ is inert in $\mathbb{Z}_p\langle \xi_1, ..., \xi_n \rangle$.

COROLLARY 3.6. *Let D be the diagram associated to an integer matrix (β_{ij}), where $\beta_{ii} = 0$ and $\beta_{ij} = \beta_{ji}$. Suppose that, for every p, D is connected* mod p. *Then the set $\{\rho_1, ..., \rho_{n-1}\}$, where ρ_i is given by* (10), *is inert in $\mathbb{Z}\langle \xi_1, ..., \xi_n \rangle$.*

Proof. Combine Proposition 3.5 with the definition of inertness in a connected graded $\mathbb{Z}$-algebra.

Combining Corollary 3.6 and Lemma 3.3, we obtain immediately

THEOREM 3.7. *Let L be a pure braid link. Suppose that, for every p, its linking diagram D is connected* mod p. *Then $\pi_1(S^3 - L)$ is a mild group.*

A special case of Theorem 3.7 arises when D has a spanning subtree consisting of edges with weight ± 1. Such links occur frequently in surgery theory (see, e.g., [7] or Sections 10D and 10E of [13]).

It would be interesting to know whether Theorem 3.7 holds for links which are not pure braid links.

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REFERENCES

1. R. ACHILLES AND L. AVRAMOV, "Relations between Properties of a Ring and Its Associated Graded Ring," Seminar Eisenbud/Singh/Vogel, Vol. 2, Teubner-Texte zur Mathematik No. 48, pp. 5–29, Teubner, Stuttgart.
2. D. ANICK, "Generic Algebras and CW Complexes," Proceedings, 1983 Conference on Algebraic Topology and K-Theory, Princeton University, to appear.
3. D. ANICK, Non-commutative graded algebras and their Hilbert series, *J. Algebra* **78** (1982), 120–140.

4. J. Birman, "Braids, Links, and Mapping Class Groups," Annals of Mathematics Study No. 82, Princeton Univ. Press, Princeton, NJ, 1974.
5. K. Brown, "Cohomology of Groups," Springer-Verlag, New York, 1982.
6. S. Eilenberg and T. Ganea, On the Lusternik–Schnirelmann category of abstract groups, *Ann. of Math.* **65** (1957), 517–518.
7. R. Fintushel and R. J. Stern, Constructing lens spaces by surgery on knots, *Math. Z.* **175** (1980), 33–51.
8. S. Halperin and J.-M. Lemaire, Suites inertes dans les algèbres de Lie graduées, *Math. Scand.*, in press.
9. A. Karrass, W. Magnus, and D. Solitar, "Combinatorial Group Theory," Dover, New York, 1976.
10. M. Kervaire, Smooth homology spheres and their fundamental groups, *Trans. Amer. Math. Soc.* **144** (1969), 67–72.
11. J. Labute, The determination of the Lie algebra associated to the lower central series of a group, *Trans. Amer. Math. Soc.* **288**, No. 1 (1985), 51–57.
12. J. Milnor and J. C. Moore, On the structure of Hopf algebras, *Ann. of Math.* **81** (1965), 211–264.
13. D. Rolfsen, "Knots and Links," Publish or Perish, Berkeley, 1976.